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Algebra (Group)

Q. 1. The intersection of any two normal subgroups of a group G is a normal subgroup of G.

Soln: Let H and K be any two normal subgroups of a group G. Since the intersection of any two subgroups of G is also a subgroup of G
$\Rightarrow H \cap K$ is also a subgroup of G.
Let $x \in G$ and $h \in H \cap K$, then $h \in H$ and $h \in K$.
Now, since H is a normal subgroup of G, so
$$x \in G,\ h \in H \Rightarrow xhx^{-1} \in H$$
Also, K is a normal subgroup of G, so
$$x \in G,\ h \in K \Rightarrow xhx^{-1} \in K$$
Thus, we have, $xhx^{-1} \in H \cap K\ \forall x \in G, \forall h \in H \cap K$
Hence, $H \cap K$ is a normal subgroup of G.

Q. 2. A subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H in G is again a right coset of H in G.

Soln: Let H be a normal subgroup of a group G. and x, y be any two elements of G.
$\Rightarrow Hx, Hy$ are any two right cosets of H in G.
We have, $(Hx)(Hy) = H(xH)y$ [By associativity]
$= H(Hx)y$ [$\because H$ is normal $\Rightarrow Hx = xH$]
$= (HH)xy$ [By associativity]
$= Hxy$ [$\because HH = H$]

Rest part of Q.2.

$x, y \in G \Rightarrow xy \in G \Rightarrow Hxy$ is a right coset of H in G, thus the product of two right cosets Hx and Hy is the right coset Hxy.

Conversely. Let H be a subgroup of G such that the product of two right cosets of H in G is again a right cosets of H in G.

i.e. $(Hx)(Hy) = Hxy, \forall x, y \in G$.

Let $x \in G, h \in H$.

Then, $xhx^{-1} = (ex)(hx^{-1}) \in (Hx)(Hx^{-1})$

Since, $e \in H = Hxx^{-1} = He = H$.

Thus, we have $xhx^{-1} \in H \ \forall x \in G, \forall h \in H$.

Therefore, H is normal subgroup of G.

Q.3 Define Normal subgroup?

Soln: Let H be a subgroup of an abelian group G, then

$Hx = xH, \forall x \in G$.

Sometimes, it is possible that G is a non-abelian group possesses a subgroup H such that $Hx = xH, \forall x \in G$. Such subgroups whose left and right cosets coincide are called normal subgroup.

In this way, we see that a subgroup H of a group G is said to be a normal subgroup of G if for every $x \in G$ and for every $h \in H$: $xhx^{-1} \in H$.